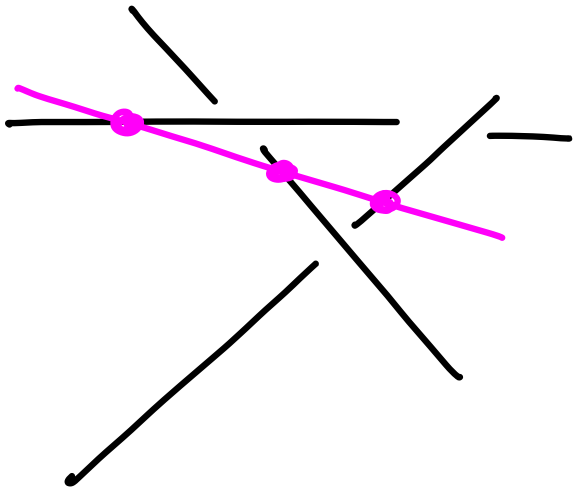
Schubert Calculus is tool to solve some algebraic geometric problems

E.g.



<u>Borel</u>: $H^*(Fl_n, \mathbb{C}) \simeq \mathbb{C}[x_1, \dots, x_n] / I_n$

$I_n = \langle e_i(x_1, \dots, x_n) \quad i = 1, \dots, n \rangle$

coinvariant algebra

elementary symm. poly

$$e_i(x_1, \dots, x_n) = \sum_{j_1 < \dots < j_i} x_{j_1} \cdots x_{j_i}$$

<u>Ex</u> $(n=3)$ consider coinvariant algebra

$$\mathbb{C}[x_1, x_2, x_3] / \langle x_1 + x_2 + x_3, x_1x_2 + x_1x_3 + x_2x_3, x_1x_2x_3 \rangle$$

Linear Basis

deg 0	1		1
deg 1	x_1	x_2	2
deg 2	x_1x_2	x_1^2	2
deg 3	$x_1x_2^2$		1.

so total dimension is 6

<u>Lemma</u> All monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$ such that such that $0 \leq a_i \leq n-i$ for $i=1,\ldots,n$ form a basis of coinvariant algebra

$$\mathbb{C}[x_1,\ldots,x_n]/I_n$$

Bernstein - Gelfand - Gelfand

defined divided difference operators

$$\partial_i f = \frac{1}{x_i - x_{i+1}} (1 - s_i) f$$

more generally, if $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition, then

$$\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$$

They showed that Schubert basis is given by $\partial_{w^{-1}w_0}(f)$ for almost any f of degree $\binom{n}{2}$.

Lascoux - Schützenberger

Schubert Polynomials

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}(x_1^{n-1} \cdots x_n^0)$$

Properties

(1) $\{\mathfrak{S}_w,\ w \in S_n\}$ mod I_n is the linear basis of coinvariant algebra

$$\mathbb{C}[x_1, \dots, x_n] / I_n$$

given by Schur classes (BGG)

(2) (non-negativity) $\mathfrak{S}_w$ have non-neg. integer coefficients.

(3) (Stability) $S_n \hookrightarrow S_{n+1}$

$$w = w_1 \cdots w_n \longrightarrow \tilde{w} = w_1 \cdots w_n (n+1)$$

Then $\mathfrak{S}_w(x_1, \dots, x_n) = \mathfrak{S}_{\tilde{w}}(x_1, \dots, x_{n+1})$

Last time we saw two definitions of Schur Polynomials

(A) Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $0 \le k \le n$

$$S_\lambda(x_1, \dots, x_k) = \mathfrak{S}_{w(\lambda)} = \partial_{w(\lambda)^{-1} w_0}(x^\delta)$$

(B) $\mu = (\mu_1, \dots, \mu_n)$

$$S_\mu(x_1, \dots, x_n) = \partial_{w_0}(x^{\lambda+\delta})$$

<u>Question</u>: How to show that (A) $\Leftrightarrow$ (B)?

<u>Ans</u>: $x^{\lambda+\delta} = \mathfrak{S}_u$ for $u \in S_m$ for some $m > n$ and use stability.

This raises the following question:

<u>Question</u>: how to see that a monomial is Schubert

$$
\begin{tikzcd}
& \mathfrak{S}_{321}=x_1^2x_2 \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
\mathfrak{S}_{231}=x_1x_2 \arrow[d, "\partial_2"] & & \mathfrak{S}_{312}=x_1^2 \arrow[d, "\partial_1"] \\
\mathfrak{S}_{213}=x_1 \arrow[dr, "\partial_1"] & & \mathfrak{S}_{132}=x_1+x_2 \arrow[dl, "\partial_2"'] \\
& \mathfrak{S}_{123}=1 &
\end{tikzcd}
$$

<u>Theorem</u> Any monomial $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ such that

(1) x^λ divides x^δ

(2) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

is a Schubert polynomial $\mathfrak{S}_w$ for some $w \in S_n$

↳ # of such monomials equal to catalan C_n

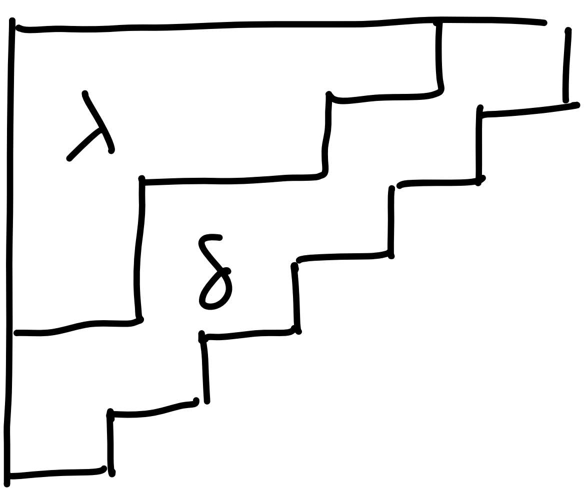
<u>Question:</u> when is $\partial_i(x_1^{\alpha_1}\cdots x_n^{\alpha_n}) =$ monomial

$$\partial_i(x_i^a x_{i+1}^b) = \begin{cases} 0 & \text{if } a=b \\ x_i^{a-1}x^b + \cdots + x_i^b x_{i+1}^{a-1} & \text{if } a>b \\ -(x_i^{b-1}x_{i+1}^a+\cdots+x_i^a x_{i+1}^{b-1}) & \text{if } a<b\end{cases}$$

<u>Ans:</u> $\alpha_i = \alpha_{i+1}+1$

<u>Claim</u> if we only allow such ∂_i we will get all Schubert monomial

<u>proof:</u> induction in a $\binom{n}{2} - |\lambda|$



If $\lambda \neq \delta$ then we can always find i s.t. $\lambda_i = \lambda_{i+1}$

we can add a box and apply induction hypothesis on μ. Then $\partial_i : x^\mu \to x^\lambda$. $\square$